

On Adiabatic Limits and Rumin's Complex

Zhong Ge *

The Fields Institute for Research in Mathematical Sciences

185 Columbia St. West, Waterloo, Ontario, N2L 5Z5

July 24, 1994

Abstract

This paper shows that when the Riemannian metric on a contact manifold is blown up along the direction orthogonal to the contact distribution, the corresponding harmonic forms rescaled and normalized in the L^2 -norms will converge to Rumin's harmonic forms. This proves a conjecture in Gromov [11]. This result can also be reformulated in terms of spectral sequences, after Forman, Mazzeo-Melrose. A key ingredient in the proof is the fact that the curvatures become unbounded in a controlled way.

*Supported by the Ministry of Colleges and Universities of Ontario and the Natural Sciences and Engineering Research Council of Canada

1 Introduction

Rumin [15] constructed a differential complex adapted to a contact distribution, for which the Laplacians are sub-elliptic operators. In this paper we show how to arrive at this complex via adiabatic limits, using the ideas of Mazzeo-Melrose [13], and Witten [18].

Beginning with Witten's work on adiabatic limits [18], there is a fair amount of work on the asymptotic behaviors of geometric-topological objects (e.g. harmonic forms, eta invariants, etc) associated with a family of Riemannian metrics on fiber bundles as the metrics become singular (see, for example, Cheeger [2]). In particular, Mazzeo-Melrose [13] studied those of harmonic forms and related them to spectral sequences (see also Forman [4]). In all these work an essential geometric assumption is that the curvatures of the metrics are uniformly bounded. In this paper we consider a different situation in which a Riemannian metric on a contact manifold is blown up along the direction orthogonal to the contact distribution. It is known that, despite that curvatures become unbounded, the Riemannian metric nevertheless converges to a Carnot-Caratheodory metric, and Gromov ([11], page 191-96) conjectured that the harmonic forms will converge to the corresponding objects associated with the Carnot-Caratheodory metrics, i.e. the Rumin's harmonic forms. In this paper we will show that this is indeed the case if the harmonic forms are rescaled and normalized in the L^2 -norms. A key ingredient in the proof is the fact that the curvatures become unbounded in a controlled way.

There is some interest to generalize Rumin's theory to more general Carnot-Caratheodory spaces (see, for example, Gromov [11]). Some preliminary results in this direction can be found in [6], [7]. The results in this paper suggest that there is probably a different approach, namely that through the study of the adiabatic limits of harmonic forms and the associated " spectral sequence " E_k^l (cf. §2, and Forman [4]). This is also related to the characteristic cohomology (cf. Bryant-Griffiths [3]) Vinogradov [17]).

The results of this paper have been announced in [9].

2 Statement of Results

Let M be a $(2m+1)$ -dimensional compact Riemannian manifold, A a contact distribution. Let B be the orthogonal distribution to A , so $TM = A \oplus B$. Write the Riemannian metric as $g = g_A \oplus g_B$. Consider a family of metrics $g_\epsilon = g_A \oplus \epsilon^{-2}g_B$. As $\epsilon \rightarrow 0$, the metric space (M, g_ϵ) converges in the sense of Gromov-Hausdorff to the Carnot-Caratheodory metric space (M, g_A) (see, for example, Fukaya [5], Ge [8], Gromov [11]), in which the distance between two points is the minimum of lengths of curves tangent to A joining the two points.

Let $\Omega^{p,q} = \Omega^p(A) \wedge \Omega^q(B)$. Decompose d into

$$d = d^{2,-1} + d^{1,0} + d^{0,1}, \quad d^{a,b} : \Omega^{p,q} \rightarrow \Omega^{p+a,q+b}.$$

Since A is contact, $d^{2,-1}$ is not zero. This is the point of departure of this paper from Mazzeo-Melrose [13].

Equip $\Omega^{p,q}$ with the metric induced from g_ϵ (still to be denoted by g_ϵ). Let Θ_ϵ be the isometry (the rescaling map)

$$\Theta_\epsilon : (\Omega(M), g_\epsilon) \rightarrow (\Omega(M), g_1).$$

Define the normalized differential $d_\epsilon = \Theta_\epsilon \circ d \circ (\Theta_\epsilon)^{-1}$, then

$$d_\epsilon = \frac{1}{\epsilon} d^{2,-1} + d^{1,0} + \epsilon d^{0,1}.$$

We will use “ $*$ ” to denote the adjoint with respect to g_1 .

Now Rumin’s complex (see Rumin [15]) can be written as

$$\mathcal{R}^k := \Omega^{k,0}/Im(d^{2,-1}), \quad k \leq m; \quad \mathcal{R}^k := \Omega^{k-1,1} \cap Ker(d^{2,-1}), \quad k \geq m+1,$$

with the induced differential

$$\begin{aligned} d_\xi &= \pi d^{1,0} : \mathcal{R}^k \rightarrow \mathcal{R}^{k+1}, & k \neq m; \\ d_\mathcal{R} &= \pi(d^{0,1} - d^{1,0}(d^{2,-1})^{-1}d^{1,0}) : \mathcal{R}^m \rightarrow \mathcal{R}^{m+1}, \end{aligned}$$

where π is the orthogonal projection $\Omega^k \rightarrow \mathcal{R}^k$. This is a sub-elliptic complex.

Theorem 2.1 *Assume (M, g) is Heisenberg (cf. §3). Suppose $\omega_\epsilon, \|\omega_\epsilon\|_{L^2} = 1$, is a d_ϵ -harmonic form,*

$$d_\epsilon \omega_\epsilon = d_\epsilon^* \omega_\epsilon = 0,$$

(i.e. $\Theta_\epsilon^{-1} \omega_\epsilon$ is a harmonic form for (M, g_ϵ)). Then as $\epsilon \rightarrow 0$, after passing to a subsequence,

$$\omega_\epsilon \rightarrow \omega_0 \neq 0 \quad \text{strongly in } L^2,$$

and ω_0 is a Rumin's harmonic form.

This result can also be reformulated in terms of spectral sequence, after Mazzeo-Melrose [13], Forman [4].

Fix a number l , one says a family of k -forms ω_ϵ is of class $O(\epsilon^l)$, i.e. $f_\delta = O(\epsilon^l)$, if $\epsilon^{-l} \|\omega_\epsilon\|_{H^1}$ is uniformly bounded, where H^1 denotes the ordinary Sobolev space. Define

$$E_l^k := \{ \omega_\epsilon \in \Omega^k, d_\epsilon \omega_\epsilon = O(\epsilon^{l-1}), d_\epsilon^* \omega_\epsilon = O(\epsilon^{l-1}), \|\omega_\epsilon\|_{L^2} = 1 \},$$

and set

$$\bar{E}_l^k := \text{linear span } \{ \text{the weakly limits of } \omega_\epsilon \text{ in } L^2 \text{ as } \epsilon \rightarrow 0, \omega_\epsilon \in E_l^k \} \cap C^\infty(\Omega^k).$$

Obviously, for each k

$$\dots \subset \bar{E}_2^k \subset \bar{E}_1^k \subset \bar{E}_0^k.$$

The following result says that most terms in the spectral sequence will degenerate at \bar{E}_2 except those of degree either m or $m+1$, which degenerate at \bar{E}_3 . This may explain why $d_{\mathcal{R}}$ is a second order operator.

Theorem 2.2 *Suppose (M, g) is Heisenberg (cf. §3).*

(1) The terms in \bar{E}^1 are

$$\bar{E}_1^k = \mathcal{R}^k.$$

(2) The terms in \bar{E}^2 are

$$\begin{aligned}\bar{E}_1^k &= \{\omega \in \mathcal{R}^k, d_\xi \omega = d_\xi^* \omega = 0\}, & k \neq m, m+1; \\ \bar{E}_2^k &= \{\omega \in \mathcal{R}^k, (d^{1,0})^* \omega = 0\}, & k = m; \\ \bar{E}_2^k &= \{\omega \in \mathcal{R}^k, d^{1,0} \omega = 0\}, & k = m+1.\end{aligned}$$

(3) The terms in \bar{E}^3 are

$$\begin{aligned}\bar{E}_3^k &= \bar{E}_2^k, & k \neq m, m+1; \\ \bar{E}_3^k &= \bar{E}_4^k = \dots = \{\omega \in \mathcal{R}^k, d_{\mathcal{R}} \omega = d_\xi^* \omega = 0\}, & k = m; \\ \bar{E}_3^k &= \bar{E}_4^k = \dots = \{\omega \in \mathcal{R}^k, d_\xi \omega = d_{\mathcal{R}}^* \omega = 0\}, & k = m+1.\end{aligned}$$

We shall use the following notations: $\|\cdot\|_{H_c^1}$ denotes the following weighted Sobolev's norm

$$\|\omega\|_{H_c^1}^2 = \int_M \sum (D_{e_i} \omega, D_{e_i} \omega) dv,$$

where e_i is an orthonormal basis for A , and $\|\omega\|_{H_c^2}$ similarly.

3 Geometry of Heisenberg Manifolds

Let v be a (local) unit tangent vector field spanning B , and ξ the contact 1-form which satisfies $\xi(v) = 1$. If the metric g_A can be written as $g_A(a, b) = d\xi(a, Jb)$, $a, b \in A$, where J is an endmorphism of A satisfying $J^2 = -Id$, then we say (M, g) is **Heisenberg**. Note that even though v is in general only locally defined, this notion is well defined. Throughout the rest of this paper we assume that (M, g) is Heisenberg.

We will use the following properties of Heisenberg manifold.

Lemma 3.1 *There is an orthonormal basis $e_1, \dots, e_m, e_{m+1} := Je_1, \dots, e_{2m} := Je_m$ for A such that*

$$\begin{aligned}[e_i, e_j] &= 0 \quad \text{mod } (A), \\ [e_i, e_{m+j}] &= \delta_{ij} v \quad \text{mod } (A), \quad 1 \leq i, j \leq m.\end{aligned}$$

Proof. This follows from the condition $g_A = d\xi(\cdot, J\cdot)$, $J^2 = -Id$.

The following property in fact holds for any contact manifold.

Lemma 3.2 *Suppose ξ is a contact 1-form and x_0 a fixed point on M , v the vector field such that $i(v)\xi = 1, i(v)d\xi = 0$. There are vector fields u_i , $i = 1, \dots, 2m+1$, $u_{2m+1} = v$, such that*

1. u_i are linearly independent at x_0 ;
2. u_i vanishes outside a small neighborhood for $i = 1, \dots, 2m$;
3. $\mathcal{L}_{u_i}\xi = \mathcal{L}_{u_i}d\xi = 0$, and \mathcal{L}_{u_i} preserves $\Omega^{k,0}$, $i = 1, \dots, 2m+1$. (Here \mathcal{L}_u is the Lie derivative in the direction u .)

Proof. First note that $\mathcal{L}_v\xi = 0$ follows from $\mathcal{L}_v = i(v)d + d i(v)$.

To choose u_1, \dots, u_{2m} , one takes a local coordinates $(x, z) \in \mathbf{R}^{2n} \times \mathbf{R}$ near x_0 such that $\xi = dz - \rho$ where ρ is a 1-form on $\mathbf{R}^{2n}\{x\}$ and $v = \partial/\partial z$. Choose $2m$ functions f_1, \dots, f_{2m} on $\mathbf{R}^{2n}\{x\}$ with linearly independent df_1, \dots, df_{2m} at x_0 such that f_i vanishes outside a neighborhood. Let H_{f_i} denote the Hamiltonian vector field of f_i with respect to $d\rho$. Define $u_i(x, z) = H_{f_i}(x) + a_i\partial/\partial z$, where a_i is determined from the equation $i(u_i)\xi = -f_i$, $i = 1, \dots, 2m$. One easily verifies that u_i thus defined satisfies all the requirements.

4 *A priori* Estimates

To prove Theorem 2.1 and Theorem 2.2, in this section we will derive some *a priori* estimates for the H_c^1 -norm of ω in terms of $(\Delta_{d_\epsilon}\omega, \omega)$ if $k \neq m, m+1$, and for the H_c^2 -norm of ω if $k = m, m+1$. As the case of $k > m$ is similar to that of $k \leq m$, we will only consider the case $k \leq m$.

We will use the following notations: If L is an operator,

$$\Delta_L := L^* L + L L^*.$$

The letter C denotes a generic positive number, M a generic constant.

4.1 The case of k -forms ($k \neq m, m+1$).

We have the following *a priori* estimates

Theorem 4.1 *For any $\omega = \alpha + \beta, \alpha \in \Omega^{k,0}, \beta \in \Omega^{k-1,1}$, we have*

$$(\Delta_{d_\epsilon} \omega, \alpha) \geq \frac{1}{\epsilon^2} \|(d^{2,-1})^* \alpha\|_{L^2}^2 + \frac{m-k}{m} C \|\alpha\|_{H_c^1}^2 + \epsilon^2 (D_v \alpha, D_v \alpha) - M(\omega, \omega),$$

$$k \leq m-1;$$

and

$$(\Delta_{d_\epsilon} \omega, \beta) \geq \frac{1}{\epsilon^2} \|(d^{2,-1}) \beta\|_{L^2}^2 + \frac{m-k+1}{m} C \|\beta\|_{H_c^1}^2 + \epsilon^2 (D_v \beta, D_v \beta) - M(\omega, \omega),$$

$$k \leq m.$$

To prove this theorem, we need a few technical results.

Lemma 4.2 *The following operator*

$$Q := (d^{0,1})^* d^{1,0} + (d^{1,0})^* d^{0,1} + d^{0,1} (d^{1,0})^* + d^{1,0} (d^{0,1})^*$$

is a first-order linear differential operator.

Proof. We only need to prove that

$$(d^{0,1})^* d^{1,0} + (d^{1,0}) (d^{0,1})^*$$

is a first order operator.

If e_i is an orthonormal basis for A , v for B , then

$$\begin{aligned} d^{1,0} &= \sum e^i \wedge D_{e_i} + \text{0-order operator} , \\ (d^{0,1})^* &= i(v)D_v + \text{0-order operator} . \end{aligned}$$

So

$$\begin{aligned} (d^{0,1})^* d^{1,0} + d^{1,0} (d^{0,1})^* &= \\ &= \sum e^i \wedge D_{e_i} i(v)D_v + i(v)D_v e^i \wedge D_{e_i} + \text{1st order operator} \\ &= \sum e^i \wedge i(v)D_{e_i}D_v + i(v)e^i \wedge D_{e_i}D_v + \text{1st order operator} \\ &= \text{1st order operator.} \end{aligned}$$

Here we have used the fact that $e^i \wedge i(v) + i(v)e^i \wedge = 0$.

Lemma 4.3 *If α, β are as in Theorem 4.1, then*

$$\begin{aligned} (\Delta_{(\epsilon^{-1}d^{2,-1}+d^{1,0})} \alpha, \alpha) &\geq \frac{1}{\epsilon^2} \|(d^{2,-1})^* \alpha\|_{L^2}^2 + C \frac{m-k}{m} \|\alpha\|_{H_c^1}^2 - M(\alpha, \alpha), \quad k \leq m-1; \\ (\Delta_{(\epsilon^{-1}d^{2,-1}+d^{1,0})} \beta, \beta) &\geq \frac{1}{\epsilon^2} \|d^{2,-1} \beta\|_{L^2}^2 + C \frac{m-k+1}{m} \|\beta\|_{H_c^1}^2 - M(\beta, \beta), \quad k \leq m. \end{aligned}$$

Proof. We shall only prove the first inequality, as the second one can be proved similarly.

By counting the types of the differential forms, one has

$$(\Delta_{(\epsilon^{-1}d^{2,-1}+d^{1,0})} \alpha, \alpha) \geq \frac{1}{\epsilon^2} \|(d^{2,1})^* \alpha\|^2 + (\Delta_{d^{1,0}} \alpha, \alpha).$$

In terms of a local orthonormal basis e_i for A , one can write

$$\begin{aligned} d^{1,0} &= \sum_{i=1}^{2m} e^i \wedge D_{e_i} + \text{0-order operator} , \\ (d^{1,0})^* &= \sum_{i=1}^{2m} i(e_i) \wedge D_{e_i} + \text{0-order operator} . \end{aligned}$$

So

$$\begin{aligned}\Delta_{d^{1,0}} &= \sum e^i \wedge i(e_j) D_{e_i} D_{e_j} + i(e_j) e^i \wedge D_{e_j} D_{e_i} + \text{1-st order operator in } e_i \\ &= -\sum D_{e_i} D_{e_i} - e^i \wedge i(e_j) (D_{e_j} D_{e_i} - D_{e_i} D_{e_j}) + \text{1-st order operators in } e_i.\end{aligned}$$

Now by Lemma 3.1, if $i > j$,

$$D_{e_i} D_{e_j} - D_{e_j} D_{e_i} = \frac{1}{m} \delta_{i-m,j} \sum_{i=1}^m (D_{e_{i+m}} D_{e_i} - D_{e_i} D_{e_{i+m}}) + \text{1st order operator in } D_{e_i},$$

while if $i < j$,

$$D_{e_i} D_{e_j} - D_{e_j} D_{e_i} = -\frac{1}{m} \delta_{i+m,j} \sum_{i=1}^m (D_{e_{i+m}} D_{e_i} - D_{e_i} D_{e_{i+m}}) + \text{1st order operator in } D_{e_i}.$$

So after an integration by parts,

$$\begin{aligned}& \int_M (e^i \wedge i(e_j) (D_{e_i} D_{e_j} \alpha - D_{e_j} D_{e_i} \alpha), \alpha) \\ &= \frac{2}{m} \int_M (i(e_i) D_{e_{j+m}} \alpha, i(e_{i+m}) D_{e_j} \alpha) - (i(e_i) D_{e_j} \alpha, i(e_{i+m}) D_{e_{j+m}} \alpha) \\ &+ \text{terms of the form } (D_{e_i} \alpha, \alpha) \\ &\leq \frac{k}{m} \|\alpha\|_{H_c^1}^2 + \text{terms of the form } (D_{e_i} \alpha, \alpha).\end{aligned}\tag{1}$$

This proves the lemma.

Proof of Theorem 4.1. We shall only prove the first inequality, as the second one can be proved similarly.

By a direct computation, one has

$$(\Delta \omega, \alpha) = (D_{(\epsilon^{-1} d^2, -1 + d^{1,0})} \omega, \alpha) + \tag{2}$$

$$+ ((d^{2,-1})^* \alpha, (d^{0,1})^* \omega) + ((d^{2,-1})^* \omega, (d^{0,1})^* \alpha) \tag{3}$$

$$+ ((d^{2,-1})^* \alpha, (d^{0,1})^* \omega) + ((d^{2,-1})^* \omega, (d^{0,1})^* \alpha) \tag{4}$$

$$+ \epsilon (Q \omega, \alpha) + \epsilon^2 (\Delta_{d^{0,1}} \omega, \alpha). \tag{5}$$

The first term (2) was considered in Lemma 4.3. By counting the types, the terms (3), (4) are zero. We only need to treat the remaining term (5).

Note that

$$(\Delta_{d^{0,1}} \omega, \alpha) = (D_v \alpha, D_v \alpha) + \int (\text{terms of the form } (D_v \omega, \alpha))$$

By the Schwartz inequality and the fact that Q (and Q^*) is a first order operator,

$$\epsilon(Q\omega, \alpha) = \epsilon(\omega, Q^* \alpha) > -\frac{\epsilon^2}{2}(\|\alpha\|_{H_c^1}^2 + \|D_v \alpha\|_{L^2}^2) - M\|\omega\|_{L^2}^2.$$

Substituting these inequalities into eq. (2)-(5), we prove the theorem.

4.2 The case of $m, (m+1)$ -forms.

If $k = m$, the estimate for the derivatives of α in Theorem 4.1 breaks down. So we need a different method to do the estimate.

Suppose $\omega = \alpha + \beta, \alpha \in \Omega^{m,0}, \beta \in \Omega^{m-1,1}$, satisfies

$$d_\epsilon \omega = \xi_1, \tag{6}$$

$$d_\epsilon^* \omega = \xi_2. \tag{7}$$

Let $\gamma_1 = D_v \alpha, \gamma_2 = D_v \beta, \gamma = \gamma_1 + \gamma_2$. We first estimate the second order derivatives of β .

Note that D_v commutes with $d^{2,-1}$ and with $(d^{2,-1})^*$ modulo zero-order operators.

Thus, taking the derivative D_v of the eqs. (6), (7), one obtains

$$\begin{aligned} d_\epsilon \gamma &= D_v \xi_1 + (\text{0-order operator in powers of } \epsilon) \omega \\ &\quad + \frac{1}{\epsilon} (\text{0-order operator}) \omega, \\ d_\epsilon^* \gamma &= D_v \xi_2 + (\text{0-order operator in powers of } \epsilon) \omega \\ &\quad + \frac{1}{\epsilon} (\text{0-order operator}) \omega. \end{aligned}$$

Integrating

$$\int_M (\Delta_{d_\epsilon} \gamma, \gamma_2).$$

as in the proof of Theorem 4.1, using the facts that ω, γ_2 are uniformly bounded in the L^2 -norm (Theorem 4.1), one obtains

Lemma 4.4 *Suppose $k = m$. If $\omega = \alpha + \beta$ satisfies eqs. (6), (7), then*

$$\|D_v \beta\|_{H_c^1}^2 \leq C \left(\frac{1}{\epsilon^2} \|\omega\|_{L^2}^2 + \|\beta\|_{L^2}^2 + \|D_v \xi\|_{L^2}^2 \right).$$

Now we estimate $\|\alpha\|_{H_c^2}$. For this purpose we need to decompose $\alpha = \alpha_1 + L\alpha_2$, since $\Omega^{m,0} = \mathcal{R}^m \oplus L(\Omega^{m-2,0})$, where $\alpha_1 \in \mathcal{R}^m$, $L : \Omega^{m-2,0} \rightarrow \Omega^m$ is defined by $L\alpha_2 = \alpha_2 \wedge d\xi$, and $\alpha_1, L\alpha_2$ are orthogonal. We will estimate the derivatives of α_1 and $L\alpha_2$ separately. We first estimate the first order derivatives of $L\alpha_2$.

Lemma 4.5

$$\|L\alpha_2\|_{H_c^1}^2 + \frac{1}{\epsilon^2} \|(d^{2,-1})^* L\alpha_2\|_{L^2}^2 \leq C(\|\xi\|_{L^2}^2 + \|\omega\|_{L^2}^2).$$

Proof. This is proved by integrating

$$\int (L\alpha_2, \Delta_{d_\epsilon} \omega)$$

as in the proof of Theorem 4.1. The key point is that $\Delta_{d^{1,0}}$ is sub-elliptic on $L(\Omega^{m-2,0})$.

In fact, one has the following estimate which improves over that in (1)

$$\begin{aligned} & \int_M (e^i \wedge i(e_j)(D_{e_i} D_{e_j} - D_{e_j} D_{e_i}) L\alpha_2, L\alpha_2) \\ &= \frac{2}{m} \int_M (i(e_i) D_{e_{j+m}} L\alpha_2, i(e_{i+m}) D_{e_j} L\alpha_2) - (i(e_i) D_{e_j} L\alpha_2, i(e_{i+m}) D_{e_{j+m}} L\alpha_2) \\ &+ \text{terms of the form } (D_{e_i} L\alpha_2, L\alpha_2) \\ &\leq \frac{m-1}{m} \|L\alpha_2\|_{H_c^1}^2 + \text{terms of the form } (D_{e_i}(L\alpha_2), L\alpha_2), \end{aligned}$$

and hence one has

$$(\Delta_{d_\epsilon} \omega, L\alpha_2) \geq \frac{1}{\epsilon^2} \|(d^{2,-1})^* L\alpha_2\|_{L^2}^2 + \frac{1}{m} C \|L\alpha_2\|_{H_c^1}^2 - M(\omega, \omega).$$

We now estimate the second order derivatives of $L\alpha_2$. Let $u = u_i$ be as in Lemma 3.2, $i = 1, 2, \dots, 2m + 1$. Take the Lie derivative of eqs. (6), (7) with respect to u , one has

$$\begin{aligned} d_\epsilon \varphi &= \mathcal{L}_u \xi_1 + (\text{0-order operator in powers of } \epsilon) \omega, \\ d_\epsilon^* \varphi &= \mathcal{L}_u \xi_2 + (\text{0-order operator in powers of } \epsilon) \omega, \end{aligned}$$

where $\varphi := \mathcal{L}_u \omega$. Note that \mathcal{L}_{u_i} preserves the decomposition $\Omega^{m,0} = \mathcal{R}^m \oplus L(\Omega^{m-2,0})$, hence, applying the same arguments as in the proof of Lemma 4.5 to the above equations, one obtains

Lemma 4.6

$$\|\mathcal{L}_{u_i} L\alpha_2\|_{H_c^1}^2 \leq C(\|\mathcal{L}_{u_i} \xi\|_{L^2}^2 + \|\xi\|_{L^2}^2 + \|\omega\|_{L^2}^2), \quad i = 1, 2, \dots, 2m + 1.$$

Then we have the following estimate on the second order derivatives of $L\alpha_2$.

Corollary 4.7

$$\sum_{i=1}^{2m} \|D_{e_i} L\alpha_2\|_{H^1}^2 \leq C(\|\xi\|_{H^1}^2 + \|\xi\|_{L^2}^2 + \|\omega\|_{L^2}^2).$$

Proof. This follows from Lemma 4.6. In fact, by Lemma 4.6, every $x_0 \in M$ has a neighborhood U such that

$$\sum_{i=1}^{2m} \|D_{e_i} L\alpha_2\|_{H^1(U)}^2 \leq C(\|\xi\|_{H^1}^2 + \|\xi\|_{L^2}^2 + \|\omega\|_{L^2}^2 + \|L\alpha_2\|_{H_c^1}).$$

Then the corollary follows from a partition of unity and Lemma 4.5.

Now we estimate the derivatives of α_1 .

Eliminating β from eq. (6) by the fact that $d^{2,-1} : \Omega^{m-2,1} \rightarrow \Omega^{m+1,0}$ is an isomorphism, one has

$$d_{\mathcal{R}} \alpha_1 = \pi((d^{2,-1})^{-1} \xi_1^1 + \frac{1}{\epsilon} \xi_1^2) - d_{\mathcal{R}} L\alpha_2, \quad (8)$$

where $\xi_1 = \xi_1^1 + \xi_1^2$, $\xi^i \in \Omega^{m-i-1, i-1}$, $i = 1, 2$. From the $(1, 0)$ -component of eq. (7) one obtains

$$d^{1,0}(d^{1,0})^* \alpha_1 = d^{1,0} (-\epsilon (d^{0,1})^* \beta + \xi_2^1) - d^{1,0}(d^{1,0})^* L\alpha_2, \quad (9)$$

where ξ_2 is decomposed into $\xi_2^1 + \xi_2^2$ as for ξ_1 . By Rumin [16], $d_{\mathcal{R}}^* d_{\mathcal{R}} + (d^{1,0}(d^{1,0})^*)^2$ is hypoelliptic on \mathcal{R}^m . (However, note that $d_{\mathcal{R}}^* d_{\mathcal{R}} + (d^{1,0}(d^{1,0})^*)^2$ is not hypoelliptic on $L(\Omega^{m-2,0})$, which is the reason why we decompose α .) Hence, from the eqs. (8)-(9), plus the following estimates

$$\begin{aligned} \|d_{\mathcal{R}} L\alpha_2\|_{L^2} &\leq C \left(\sum_{i=1}^{2m} \|D_{e_i} L\alpha_2\|_{H^1} + \|L\alpha_2\|_{L^2} \right), \\ \|d^{1,0}(d^{1,0})^* L\alpha_2\|_{L^2} &\leq C \left(\sum_{i=1}^{2m} \|D_{e_i} L\alpha_2\|_{H^1} + \|L\alpha_2\|_{L^2} \right), \end{aligned}$$

which can be controlled by using Corollary 4.7, and

$$\|d^{1,0}(d^{0,1})^* \beta\|_{L^2} \leq C (\|D_v \beta\|_{H_c^1}^2 + \|\beta\|_{H_c^1}^2 + \|\beta\|_{L^2}^2),$$

which can be controlled by using Lemma 4.4, one obtains

Theorem 4.8 *If $\omega = \alpha + \beta$ satisfies eqs. (6), (7), then*

$$\|\alpha\|_{H_c^2}^2 \leq C (\|\omega\|_{L^2}^2 + \|\xi\|_{H^1}^2 + \frac{1}{\epsilon^2} \|\xi\|_{L^2}^2).$$

5 Proof of Theorem 2.1.

Much of the proof depends on the properties of $d^{2,-1}$, which we list now. These properties follow from a straight forward computation.

Lemma 5.1 *(1) $d^{2,-1} : \Omega^{k-2,1} \rightarrow \Omega^{k,0}$ is an injection for $k \leq m-1$, an isomorphism for $k = m$.*

(2) $(d^{2,-1})^ : \Omega^{k,0} \rightarrow \Omega^{k-2,1}$ is an injection for $k \geq m+2$, an isomorphism for $k = m+1$.*

Let $\omega = \alpha + \beta$, $\alpha \in \Omega^{k,0}$, $\beta \in \Omega^{k-1,1}$ be as in Theorem 2.1. Then $d_\epsilon \omega = (d_\epsilon)^* \omega = 0$ is equivalent to

$$\frac{1}{\epsilon} d^{2,-1} \beta + d^{1,0} \alpha = 0, \quad (10)$$

$$d^{1,0} \beta + \epsilon d^{0,1} \alpha = 0, \quad (11)$$

$$\frac{1}{\epsilon} (d^{2,-1})^* \alpha + (d^{1,0})^* \beta = 0, \quad (12)$$

$$(d^{1,0})^* \alpha + \epsilon (d^{0,1})^* \beta = 0. \quad (13)$$

Lemma 5.2 *Suppose \mathcal{Q} is a first-order differential operator. If $\|\omega_\epsilon\|_{L^2}$, $\|\epsilon \mathcal{Q} \omega_\epsilon\|_{L^2}$ are uniformly bounded, then*

$$\epsilon \mathcal{Q} \omega_\epsilon \rightarrow 0, \quad \text{weakly in } L^2.$$

Proof. We may choose a sub-sequence such that

$$\epsilon \mathcal{Q} \omega_\epsilon \rightarrow \gamma, \quad \text{weakly in } L^2.$$

We need only to prove $\gamma = 0$. Assume without loss of generality that $\omega \rightarrow a$ weakly in L^2 . Now choose a smooth k -form μ , then

$$(\gamma, \mu)_{L^2} = \lim_{\epsilon \rightarrow 0} (\epsilon \mathcal{Q} \omega_\epsilon, \mu)_{L^2} = \lim_{\epsilon \rightarrow 0} \epsilon (\omega_\epsilon, (\mathcal{Q})^* \mu)_{L^2} = 0,$$

so $\gamma = 0$.

Proof of Theorem 2.1. We will only prove the theorem for the case $k \leq m$, as the case $k > m$ is similar. We divide the proof into two cases: one for $k \leq m - 1$, the other $k = m$.

(1) $k \leq m - 1$. By Theorem 4.1 we observe that both α and β are uniformly bounded in H_c^1 , and $\epsilon^{-1} d^{2,-1} \beta$ is uniformly bounded in L^2 . By Lemma 5.1, this implies that α converges to α_0 , after passing to a subsequence if necessary, and β to 0 strongly in L^2 .

By Theorem 4.1, $\epsilon \|d^{0,1}\beta\|_{L^2}$ is bounded. Then by Lemma 5.2, $\epsilon d^{0,1}\beta \rightarrow 0$ weakly in L^2 . Then, from eq. (11), it follows that α_0 satisfies

$$d^{1,0}\alpha_0 = 0$$

in the weak sense that for any $\mu \in H_c^1$, $(\alpha_0, (d^{1,0})^* \mu)_{L^2} = 0$.

Similarly, from eq. (12) and Lemma 5.2 we have

$$(d^{1,0})^* \alpha_0 = 0, \quad (d^{2,-1})^* \alpha_0 = 0$$

in the weak sense. Now the theory of sub-elliptic operators implies that α_0 is smooth and satisfies the Rumin's Laplacian (cf. Helffer-Nourrigat [12]).

To conclude the proof, we note that $\alpha_0 \neq 0$, as ω converges to α_0 strongly in L^2 , and $\|\omega\|_{L^2} = 1$. This proves the theorem for $k < m$.

(2) If $k = m$, then it follows from Theorem 4.1, Theorem 4.8, that $\|\alpha\|_{H_c^2}$, $\|\beta\|_{H_c^1}$ are uniformly bounded. Moreover, as in the case $k \leq m - 1$, $\beta \rightarrow 0$ weakly in H_c^1 . We may choose a subsequence of α such that $\alpha \rightarrow \alpha_0$ weakly in H_c^2 .

It follows from eqs. (10), (11), that

$$d_{\mathcal{R}}\alpha = 0.$$

Hence α_0 also satisfies the above equation in the weak sense that for any $\mu \in H_c^2$, $(\alpha_0, (d_{\mathcal{R}})^* \mu)_{L^2} = 0$.

Now that $\epsilon (d^{1,0})^* \beta, \epsilon (d^{0,1})^* \beta$ are uniformly bounded in L^2 , by Lemma 5.2, $\epsilon (d^{1,0})^* \beta \rightarrow 0, \epsilon (d^{0,1})^* \beta \rightarrow 0$ weakly in L^2 . Then it follows from eqs. (12), (13) that

$$(d^{1,0})^* \alpha_0 = 0, \quad (d^{2,-1})^* \alpha_0 = 0$$

in the weak sense. Thus α_0 is a Rumin's harmonic form.

At last note that since $\omega \rightarrow \alpha_0$ strongly in L^2 , $\alpha_0 \neq 0$. This proves the theorem.

6 Proof of Theorem 2.2.

As before, we only consider the case $k \leq m$, as the case $k > m$ is similar.

First note that the equations $d_\epsilon \omega = \xi_1, d_\epsilon \omega = \xi_2$, are equivalent to

$$\frac{1}{\epsilon} d^{2,-1} \beta + d^{1,0} \alpha = \xi_1^1, \quad (14)$$

$$d^{1,0} \beta + \epsilon d^{0,1} \alpha = \xi_1^2, \quad (15)$$

$$\frac{1}{\epsilon} (d^{2,-1})^* \alpha + (d^{1,0})^* \beta = \xi_2^1, \quad (16)$$

$$(d^{1,0})^* \alpha + \epsilon (d^{0,1})^* \beta = \xi_2^2, \quad (17)$$

where $\xi_i = \xi_i^1 + \xi_i^2, \xi_i^j \in \Omega^{*,j-1}$.

(1) $l = 1$. Let $\omega_\epsilon = \alpha_\epsilon + \beta_\epsilon \in E_k^1$. We will study the limit of ω_ϵ as $\epsilon \rightarrow 0$.

Suppose $k \leq m - 1$. By Theorem 4.1, we see that $\|\omega_\epsilon\|_{H_c^1}, \|\epsilon^{-1} (d^{2,-1})^* \alpha_\epsilon\|_{L^2}$ and $\|\epsilon^{-1} d^{2,-1} \beta_\epsilon\|_{L^2}$ are uniformly bounded. As in the proof of Theorem 2.1, this implies that after passing to a subsequence, $\omega \rightarrow \omega_0$ weakly in H_c^1 , where $\omega_0 \in \Omega^{k,0} \cap \text{Ker}((d^{2,-1})^*) = \mathcal{R}^k$. So $\bar{E}_k^1 \subset \mathcal{R}^k$ for $k \leq m - 1$.

We will prove that this inclusion relation also holds for $k = m$. If $k = m$, it follows from Theorem 4.1 that $\|\beta\|_{H_c^1}, \|\epsilon^{-1} d^{2,-1} \beta_\epsilon\|_{L^2}$ are uniformly bounded. So $\beta_\epsilon \rightarrow 0$ weakly in H_c^1 by Lemma 5.1.

Also, since $\|\epsilon (d^{1,0})^* \beta_\epsilon\|_{L^2}$ is uniformly bounded (Theorem 4.1), by Lemma 5.2, $\epsilon (d^{1,0})^* \beta_\epsilon \rightarrow 0$ weakly in L^2 . It follows from eq. (16) that the weak limit ω_0 of ω_ϵ in L^2 satisfies

$$(d^{2,-1})^* \omega_0 = 0.$$

So $\bar{E}_m^1 \subset \mathcal{R}^m$.

Conversely, we will prove $\mathcal{R}^k \subset \bar{E}_k^1$ for $k \leq m$. This follows from the fact that if $\alpha_0 \in \Omega^{k,0} / \text{Im}(d^{2,-1})$, $\|\alpha_0\|_{L^2} = 1$, then obviously $\alpha_0 \in E_k^1$ and hence $\alpha_0 \in \bar{E}_k^1$.

(ii) $l = 2$. The proof for the case $k \neq m$ is similar to that of Theorem 2.1 and will be omitted here.

Consider the case $k = m$. By Theorem 4.1, Theorem 4.8, $\|\alpha_\epsilon\|_{H_c^2}, \|\beta_\epsilon\|_{H_c^1}$ and $\|\epsilon^{-1} d^{2,-1} \beta_\epsilon\|_{L^2}$ are uniformly bounded. Let α_0, β_0 be the weak limit of $\alpha_\epsilon, \beta_\epsilon$ in H_c^2 ,

H_c^1 respectively. First note that, as in the case $l = 1$, the weak limit $\omega_0 = \alpha_0 + \beta_0 \in \mathcal{R}^m$.

Moreover, by Lemma 5.2, $\epsilon(d^{0,1})^*\beta_\epsilon \rightarrow 0$ weakly in L^2 . So it follows from eq. (17) that $(d^{1,0})^*\alpha_0 = 0$. Thus $\bar{E}_m^2 \subset \mathcal{R}^m \cap \text{Ker}((d^{1,0})^*)$.

Conversely, if $\alpha_0 \in \mathcal{R}^m \cap \text{Ker}((d^{1,0})^*)$, then obviously $\alpha_0 \in \bar{E}_m^2$. So $\mathcal{R}^m \cap \text{Ker}((d^{1,0})^*) \subset \bar{E}_m^2$. This proves the case $l = 2$.

(iii) $l = 3$. The proof in this case is similar to that of Theorem 2.1 and will be omitted.

References

References

- [1] J. M. Bismut and D. S. Freed *The analysis of elliptic families, I. Metrics and connections on determinant bundles*, Comm. Math. Phys., 106(1986), pp. 156-176. *II. Dirac operators, eta invariants and the holonomy theorem*, 107(1986), pp. 103-163.
- [2] J. Cheeger , *Eta invariants, the adiabatic approximation and conical singularities*, J. Diff. Geometry, 26(1987), pp. 175-221.
- [3] R. L. Bryant and P. A. Griffith, *Characteristic cohomology of differential systems (I): general theory* , preprint.
- [4] R. Forman, *Spectral sequences and adiabatic limits*, preprint.
- [5] K. Fukaya, *Collapsing of Riemannian manifolds and eigenvalues of Laplacian operators*, Invent. Math., 87(1987), pp. 161-207.
- [6] Z. Ge, *A non-holonomic Hodge theory for sub-Riemannian metrics*, preprint.

- [7] Z. Ge, *Betti numbers, characteristic classes and sub-Riemannian geometry*, Illinois J. of Mathematics, 36(1992), pp. 372-403.
- [8] Z. Ge, *Collapsing Riemannian metrics to Carnot-Caratheodory metrics and Laplacians to sub-Laplacians*, Canadian Journal of Math., 45(1993), pp. 537-553.
- [9] Z. Ge, *Adiabatic limits and Rumin's complex*, to appear in C. R. Acad. Sci. Paris Sèr. I Math.
- [10] V. Ginzburg, *Closed characteristics of closed two-forms*, Ph. D. thesis, Berkeley, 1990.
- [11] M. Gromov, *Carnot-Caratheodory spaces seen from within*, IHES preprint (221 pages), 1994.
- [12] B. Helffer and J. Nourrigat, *Hypoellipticité maximale pour des opérateurs polynômes de champs de vecteurs*, Progress in Math., 58, Birkhäuser, 1985.
- [13] R. Mazzeo and R. Melrose, *The adiabatic limit, Hodge cohomology and Leary's spectral sequence for a fibration*, J. of Diff. Geometry, 31(1991), pp. 185-213.
- [14] P. Pansu, *Differential forms and connections adapted to a contact structure, after M. Rumin*, preprint, 1990.
- [15] M. Rumin, *Un complexe de formes différentielle sur les variétés de contact*, C. R. Acad. Sci. Paris, t. 310, 1990, Serie I., p401-404.
- [16] M. Rumin, *Formes différentielles sur les variétés de contact*, J. of Differential Geometry, 39(1994), pp. 281-330.
- [17] A. M. Vinogradov, *Geometry of non-linear differential equations*, J. Soviet Math., 17(1981), pp. 1624-1649.

- [18] E. Witten *Global gravitational anomalies*, Comm. Math. Phys., 100(1985), pp. 197-229.